Inequalities of Miyaoka-Yau-type in Intermediate Kodaira Dimension

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Recall: $\left.\begin{array}{l}\bullet\ X^n \text{ a compact complex manifold} \\ \bullet\ E \text{ a } \mathbb{C}\text{-vector bundle}\end{array}\right\} \rightsquigarrow c_j(E) := \left(\frac{1}{2\pi i}\right)^j \mathrm{Tr}(F_\nabla \wedge \dots \wedge F_\nabla) \in H^{2j}_{dR}(X, \mathbb{C})$

∇ a connection in E

$F_\nabla :=$ the curvature

Ex: $\bullet\ \mathrm{rk}\, E = 1 \rightsquigarrow c_1(E) = PD[\mathcal{Z}(s)]$ $\quad s$ a general section of E

$\bullet\ E = T_X \rightsquigarrow c_i(X) = c_i(T_X) = (-1)^i c_i(\underset{(T_X)^*}{\underset{\shortparallel}{\Omega_X}})$, $c_n(X) = \chi^{top}(X)$

Thm: (..., Miyaoka, Yau)

Let X be a minimal (non-uniruled) smooth proj. surface $/\mathbb{C}$. Then

not the blow-up of another smooth surface

$$2(\nu+1)\, c_2(X) - \nu \underbrace{c_1(X)^2}_{\geq 0} \geq 0 \qquad \Longrightarrow \chi^{top}(X) \geq 0$$

Moreover, $3c_2(X) = c_1(X)^2$ iff (up to finite étale cover)

(1) $X = A$ an Abelian surface $\quad (\nu = 0)$

(2) $X \xrightarrow{f} C$ admits a smooth elliptic fibration / a curve of genus $g(C) \geq 2$ $\quad (\nu = 1)$

(3) $X \cong \mathbb{B}^2/\Lambda$, $\mathbb{B}^2 = \{z \in \mathbb{C}^2 \mid \|z\| < 1\}$, $\quad (\nu = 2)$

$\Lambda \leq \mathrm{Aut}(\mathbb{B}^2) = PU(2,1)$

$$\nu := \nu(X) = \max\{k \in \mathbb{Z} \mid c_1(X)^k \neq 0 \in H^*(X, \mathbb{C})\}$$

Q: What about $\dim X \geq$? What about singularities?

Thm: (many authors...) (e.g. fin. quot. singularities)

Let X^n be a log terminal proj. variety $/\mathbb{C}$ s.t. K_X is nef. ($K_X \cdot C \geq 0$ $\forall C \subseteq X$ curve) $\nu := \nu(X)$. Then $\exists \varepsilon_0 > 0$:

$$(*) \qquad \left(2(\nu+1)c_2(X) - \nu c_1(X)^2\right)(K_X + \varepsilon H)^{n-2} \geq 0 \quad \forall \begin{array}{l} H \text{ ample Cartier} \\ 0 < \varepsilon < \varepsilon_0 \end{array}$$

Moreover, equality holds iff (up to fin. quasi-étale cover)

(1) $X = A$ an Abelian variety $\qquad (\nu = 0)$

(2) $X \xrightarrow{f} C$ admits a smooth Abelian fibration / curve of genus ≥ 2 $\qquad (\nu = 1)$

(3) $X = A \times B$ $\quad$ A an Abelian variety, $\underset{\text{a ball quotient}}{B \cong \mathbb{B}^\nu/\Lambda}$ $\qquad (\nu \geq 2)$

In particular, K_X is semiample.

Rmk: In case (2), f need not be isotrivial

What about ε?

Q: $\underset{\sum a_i X^i}{P_1}, \underset{\sum b_i X^i}{P_2} \in \mathbb{R}[X]$:

$$\begin{pmatrix} \exists \varepsilon_0 > 0 \\ P_1(\varepsilon) < P_2(\varepsilon) \\ \forall 0 < \varepsilon < \varepsilon_0 \end{pmatrix} \iff \begin{pmatrix} \exists k \in \mathbb{Z} \\ a_0 = b_0 \\ a_1 = b_1 \\ \vdots \\ a_{k-1} = b_{k-1} \\ a_k < b_k \end{pmatrix}$$

$\nu = \dim X = n$ $\rightsquigarrow$

$$\left(2(n+1)c_2(X) - n c_1(X)^2\right)K_X^{n-2} \geq 0 \qquad \text{(GKPT)}$$

If equality holds then $\left(2(n+1)c_2(X) - n c_1(X)^2\right)K_X^{n-3} \cdot H \geq 0$ & $X \xrightarrow[\text{in codim two}]{\text{iso}} \mathbb{B}^n/\Lambda$

If equality holds then $\left(2(n+1)c_2(X) - n c_1(X)^2\right)K_X^{n-4} H^2 \geq 0$ & $X \xrightarrow[\text{codim 3}]{\text{iso in}} \mathbb{B}^n/\Lambda$

$\vdots \qquad\qquad \vdots$

Rmk: $\exists X$ log terminal proj. variety, K_X nef. $\nu(X) = n$,

$$\left(2(n+1)c_2(X) - n c_1(X)^2\right)K_X^{n-2} \quad \text{but} \quad X \not\cong \mathbb{B}^n/\Lambda$$

Who proved what?

- $\dim X = 3$: Peternell-Wilson

ν	Inequality	Characterisation of Eq. case
0	Miyaoka	Yau, ..., Lu-Taji
1	Miyaoka	Iwai-Matsumura-M.
2	Miyaoka	Iwai-Matsumura-M.
$\vdots$	in my thesis	in my thesis
$\dim X$	Yau, .., Guenancia-Taji	Yau, .., Greb-Kebekus-Peternell-Taji

- Hao-Schreider: First results when $\nu(X) \notin \{0, \dim X\}$

How to proof inequalities for $c_2(X)$? (à la Miyaoka)

Set-up: • X log terminal proj. variety, $H_1, \dots, H_{n-1}$ ample divisors

• $(\mathcal{E}, \vartheta)$ a reflexive coherent Higgs sheaf on X, $\vartheta : \mathcal{E} \otimes \mathcal{T}_X \to \mathcal{E}$
$\mathcal{O}_X$-linear + $[\vartheta, \vartheta] = 0$

Assume: $\mathcal{E}$ is generically semipositive, i.e.

$$\forall\, \mathcal{E} \twoheadrightarrow Q : \quad c_1(Q) H_1 \cdots H_{n-1} \geq 0 \iff \begin{array}{l} c_1(\mathcal{F}) H_1 \cdots H_{n-1} \\ \leq c_1(\mathcal{E}) H_1 \cdots H_{n-1} \end{array} \ \forall \mathcal{F} \subseteq \mathcal{E}$$

Ex: $\mathcal{E} := \Omega_X^{[1]} := (\Omega_X^1)^{**}$ ← gen. semipos. if K_X is nef (Campana-Păun)

Thm: (Miyaoka, Langer)
Assume that $c_1(\mathcal{E})$ is nef. Then

$$2c_2(\mathcal{E}) H_1 \cdots H_{n-2} \geq c_1(\mathcal{E})^2 H_1 \cdots H_{n-2} - \sup_{\substack{\mathcal{F} \subseteq \mathcal{E} \\ \text{Higgs subsheaf}}} \Big\{ \underbrace{\frac{c_1(\mathcal{F}) c_1(\mathcal{E}) H_1 \cdots H_{n-2}}{\operatorname{rk} \mathcal{F}}}_{=: \mu(\mathcal{F})\ \text{"slope"}} \Big\}$$

Slogan: Upper bounds for $\mu(\mathcal{F})$, $\mathcal{F} \subseteq \mathcal{E}$
$\Rightarrow$ Lower bounds for $c_2(\mathcal{E})$!

Ex: • $\mathcal{E}$ gen. semipos. $\underset{\text{Miyaoka}}{\Rightarrow}$ $\mu(\mathcal{F}) \leq \frac{c_1(\mathcal{E})^2 H_1 \cdots H_{n-2}}{\operatorname{rk}\mathcal{F}} \leq c_1(\mathcal{E})^2 H_1 \cdots H_{n-2}$

$\underset{\text{Langer}}{\Rightarrow}$ $c_2(\mathcal{E}) H_1 \cdots H_{n-2}$

• If $\mathcal{E}$ is "semistable" i.e. $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ $\forall \mathcal{F} \subseteq \mathcal{E}$ then

$$\Rightarrow \left(2\operatorname{rk}\mathcal{E}\, c_2(\mathcal{E}) - (\operatorname{rk}\mathcal{E}-1)\, c_1(\mathcal{E})^2\right) H_1 \cdots H_{n-2} \geq 0$$

"Bogomolov-Gieseker inequality"

Ex: X^n log terminal, K_X nef, $\mathcal{E} := \Omega_X^{[1]}$

∘ $c_2(X) H_1 \cdots H_{n-2} \geq 0$

∘ If $\nu(X) = n$ $\underset{\text{Taji}}{\overset{\text{Guenancia-}}{\Longrightarrow}}$ $\Omega_X^{[1]}$ is K_X^{n-1}-semistable $\Rightarrow$ $(\Omega_X \oplus \mathcal{O}_X, \vartheta)$ is K_X^{n-1}-semistable

$\underset{\text{Gieseker}}{\overset{\text{Bogomolov-}}{\Longrightarrow}}$ $\left(2(n+1)c_2(X) - n\, c_1(X)^2\right) K_X^{n-2} \geq 0$

Observation: (Simpson) $\vartheta : (\Omega_X^{[1]} \oplus \mathcal{O}_X) \otimes \mathcal{T}_X \longrightarrow \Omega_X^{[1]} \oplus \mathcal{O}_X$ Higgs sheaf

$(\eta, f) \otimes v \longmapsto (0, \eta(v))$

Easy fact: $\Omega_X^{[1]}$ semistable $\Rightarrow$ $(\Omega_X^{[1]} \oplus \mathcal{O}_X, \vartheta)$ is semistable

Q: X log terminal proj. variety, $\nu(X) = \nu$.

$\forall \mathcal{F} \subseteq \Omega_X^{[1]}$: $\frac{c_1(\mathcal{F}) K_X^{\nu-1} H^{n-\nu}}{\operatorname{rk}\mathcal{F}} \overset{?}{\leq} \frac{K_X^{\nu} H^{n-\nu}}{\nu}$

If answer is YES then $\left(2(\nu+1)\, c_2(X) - \nu c_1(X)^2\right) K_X^{\nu-2} H^{n-\nu} \geq 0$

<u>Known cases:</u> ◦ $\nu = 0, 1$: Generic Semipos. ✓

◦ $\nu = \dim X$: Guenancia-Taji

◦ Any ν if $H = K_X + \varepsilon H_0$ for small ε